10/6/21

GOAL: Extend composition to Calc 3

Calc I: $\mathbb{R} \xrightarrow{f} \mathbb{R} \xrightarrow{g} \mathbb{R}$ $= (g \cdot f)(x)$

Calc III $\mathbb{R}^n \xrightarrow{f} \mathbb{R}$ $\mathbb{R}^n \xrightarrow{g} \mathbb{R}$

Given $f: D \subseteq \mathbb{R}^n \to \mathbb{R}$ $f(x_1, x_2 \ldots x_n)$
letting $g_i(t_1, t_2 \ldots t_n)$ for $1 \leq i \leq n$

WE CAN DEFINE COMPOSITION F OF G_is

$F(G_1(t_1, t_2, t_3 \ldots t_n), g_2(t_1 \ldots t_n) \ldots g_n(t_1, \ldots$

EX: Suppose $f(x, y, z) = \cos(x+y) z^2 + 3$

$x(s,t) = s + t$, $y(s,t) = s \cdot t$ $z(s,t) = \cos(s)$

$f(x(s,t), y(s,t), z(s,t)) = \cos(s+t+st) \cos(s)^2 + 3$

$\mathbb{R}^k \xrightarrow{g_1} \mathbb{R}^3 \xrightarrow{f} \mathbb{R}$
$\mathbb{R}^k \xrightarrow{g_2}$
$\mathbb{R}^k \xrightarrow{g_3}$

$\mathbb{R}^k \xrightarrow{g} \mathbb{R}^n \xrightarrow{f} \mathbb{R}$

Now to extend chain rule to Calc 3

Setup: Let $f(x,y)$ and $x(t), y(t)$ be differentiable functions.

Def: A function $f: D \subseteq \mathbb{R}^n \to \mathbb{R}$ is differentiable at p when f is "well aproximated" by a tangent hyperplane at p.

As you get closer to p the error by APROXIMATING with the HYPER PLANE GOES TO 0.

Now GIVEN f, x, y with $p = (a,b)$

$$f(x,y) = f(a,b) + ((f_x(a,b) + \varepsilon_x(x,y))(x-a)) + (f_y(a,b) + \varepsilon_y(x,y))(y-b)$$

POINT

Y PARTIAL

ERROR TERM FOR Y

$(\varepsilon_x, \varepsilon_y) \to (0,0)$ as $(x,y) \to (a,b)$

CHANGE IN INPUT

$$\therefore f(x,y) - f(a,b) = (f_x(a,b)(x-a) + f_y(a,b)(y-b)) + (\varepsilon_x(x-a) + \varepsilon_y(y-b))$$

Chose a time α where $(x(\alpha), y(\alpha)) = p = (a,b)$

Substituting into the function we obtain

$$f(x(t), y(t)) - f(x(\alpha), y(\alpha)) = f_x(x(\alpha), y(\alpha))(x(t) - x(\alpha)) + f_y(x(\alpha), y(\alpha))(y(t) - y(\alpha)) + \varepsilon_x(x(t) - x(\alpha)) + \varepsilon_y(y(t) - y(\alpha))$$

For each $t \neq \alpha$ we divide by $t - \alpha$ to obtain

* let this expression be represented by Ω

$$\frac{f(x(t),y(t)) - f(x(\alpha),y(\alpha))}{t-\alpha} = f_x(x(\alpha),y(\alpha))\left(\frac{x(t)-x(\alpha)}{t-\alpha}\right) + f_y(x(\alpha),y(\alpha))\left(\frac{y(t)-y(\alpha)}{t-\alpha}\right) + \varepsilon_x\left(\frac{x(t)-x(\alpha)}{t-\alpha}\right) + \varepsilon_y\left(\frac{y(t)-y(\alpha)}{t-\alpha}\right)$$

Now limit it as $t \to \alpha$ we can understand the above as derivatives. Notice now this is just a case 1 derivative WITH RESPECT TO T

$$\lim_{t\to\alpha} \frac{f(x(t),y(t)) - f(x(\alpha),y(\alpha))}{t-\alpha}$$

$$\frac{d}{dx}\left[f(x(t),y(t))\right] = f_x(x(\alpha),y(\alpha))\lim_{t\to\alpha}\frac{x(t)-x(\alpha)}{t-\alpha} + f_y(x(\alpha),y(\alpha))\lim_{t\to\alpha}\frac{y(t)-y(\alpha)}{t-\alpha}$$

$$+\lim_{t\to\alpha}\varepsilon_x \cdot \lim_{t\to\alpha}\frac{x(t)-x(\alpha)}{t-\alpha}$$

$$+\lim_{t\to\alpha}\varepsilon_y \cdot \lim_{t\to\alpha}\frac{y(t)-y(\alpha)}{t-\alpha}$$

$$= f_x(x(\alpha),y(\alpha))x'(\alpha) + f_y(x(\alpha),y(\alpha))y'(\alpha)$$

$$+\lim_{t\to\alpha}\varepsilon_x \cdot x'(\alpha) + \lim_{t\to\alpha}\varepsilon_y(y'(\alpha)$$

$\lim_{t\to\alpha}\varepsilon_x = 0$ by our definition

$$\frac{d}{dx}f(x(t),y(t))\Big|_{\rho} = f_x(x(\alpha),y(\alpha))x'(\alpha) + f_y(x(\alpha),y(\alpha))y'$$

Propostn Multivariate Chain Rule

Let $f(x_1 .. x_n)$ and $x_i(t_1, t_k)$ be differentiable for $1 \le i \le n$ then

*CAN'T cancel out ∂x_i

$$\frac{\partial f}{\partial t_i} = \frac{\partial f}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_j} + \frac{\partial f}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_j} \dots + \frac{\partial f}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_j}$$

for all $1 \le j \le k$

Partial Derivative of f with respect to the direction of x_1

Partial derivate

PARTIAL DERIVATIVE OF THE FUNCTION FOR x_1 with Respect to a CERTAIN DIRECTION t_j. For Single Variable functions we can only differentiate with respe to one variable. Now we can do it with any t_j

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EX) COMPUTE $\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}$ $\quad f(x,y) = e^x \sin(y)$

$x = st^2 \quad y = s^2 t$

Solution 1: No chain rule. $f(x,y) = f(st^2, s^2 t)$

$= e^{st^2} \sin(s^2 t)$ $\quad$ VIA PRODUCT RULE

$$\frac{\partial f}{\partial s} = t^2 e^{st^2} \sin(s^2 t) + 2st \cos(s^2 t) e^{st^2}$$

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial t}(e^{st^2}) \sin(s^2 t) + \frac{\partial}{\partial t}(\sin(s^2 t)) e^{st^2}$$

$$2ts e^{st^2} \sin^{s^2 t} + s^2 \cos(s^2 t) e^{st^2}$$

* USING CHAIN RULE ONLY USUALLY MAKES THINGS EASIER

Solution 2: With Chain rule

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s}$$

and

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}$$

$$\frac{\partial f}{\partial x} = e^{x}\sin(y) = e^{st^2}\sin(ts^2) \qquad \frac{\partial f}{\partial y} = \cos(y)e^{x} = \cos(ts^2)e^{t^2s}$$

$$\frac{dx}{ds} = t^2 \qquad \frac{dx}{dt} = 2st \qquad \frac{dy}{ds} = 2st \qquad \frac{dy}{dt} = s^2$$

$$\therefore \frac{df}{ds} = e^{st^2}\sin(ts^2)\cdot t^2 + e^{st^2}\cos(s^2t)\cdot 2st$$

$$\frac{\partial f}{\partial t} = e^{st^2}\sin(ts^2)\cdot 2st + e^{st^2}\cos(s^2t)\cdot s^2$$

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let $f(x,y,z) = x^4y + y^2z^3$

$x = rse^t \qquad y = rs^2e^{-t} \qquad z = r^2s\sin(t)$

Compute $\frac{\partial f}{\partial r} \quad \frac{\partial f}{\partial s} \quad \frac{\partial f}{\partial t}$

DO AT HOME

Recall from Calc 1

Given an equation with both x and y

$(x-y)^2 = x + y^2$, you could compute implicit derivatives of both sides

Implicit Function THEOREM? IFT

Let $F(x_1, x_2, \ldots, x_n)$, differentiable, and $\frac{\partial F}{\partial x_i}$ be continuous on a disk about point p.

and $\left.\frac{\partial F}{\partial x_n}\right|_{\vec{p}} \neq 0$ and $F(\vec{p}) = 0$

Back to before we can think of

$(x-y^2) = x+y^2 \rightarrow \underbrace{(x-y)^2 - x - y^2}_{F(\vec{p})} = 0$

Then $x_n = F(x_1, \ldots, x_{n-1})$, for a point near p, is a separate function of the other variables

$\frac{\partial f}{\partial x_i} = -\frac{\partial F}{\partial x_i} \Big/ \frac{\partial F}{\partial x_n}$